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String order and hidden topological symmetry in the $SO(2n+1)$ symmetric matrix product states

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Abstract

We have introduced a class of exactly soluble Hamiltonian with either $SO(2n+1)$ or $SU(2)$ symmetry, whose ground states are the $SO(2n+1)$ symmetric matrix product states. The hidden topological order in these states can be fully identified and characterized by a set of nonlocal string order parameters. The Hamiltonian possesses a hidden $(Z_2 \times Z_2)^n$ topological symmetry. The breaking of this hidden symmetry leads to 4^n degenerate ground states with disentangled edge states in an open chain system. Such matrix product states can be regarded as cluster states, applicable to measurement-based quantum computation.

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Quantum spin systems have shown many fascinating phenomena and stimulated great interest in the past decades. Based on semiclassical argument, Haldane predicted that there is a finite excitation gap in the ground state of an integer antiferromagnetic Heisenberg spin chain [1]. This intriguing feature of quantum spin chains results from the breaking of a hidden topological symmetry embedded in the valence bond solid state proposed by Affleck, Kennedy, Lieb and Tasaki (AKLT) [2]. The valence bond solid is a matrix product state in one dimension. It shows a striking analogy to the Laughlin ground state for the fractional quantum Hall effect [3, 4]. To characterize this topological symmetry, a set of nonlocal string order parameters were introduced [5, 6]. These string order parameters provide a faithful quantification of the hidden antiferromagnetic order of the $S = 1$ Heisenberg model. Associated with these order parameters, a nonlocal unitary transformation can be constructed to expose explicitly the $Z_2 \times Z_2$ symmetry of the Hamiltonian [6–8]. However, a nonlocal string order parameter that reflects correctly the hidden $Z_{S+1} \times Z_{S+1}$ topological symmetry of the higher-S valence bond solid has not been found [9].

In this paper, we introduce a novel matrix product state with $SO(2n+1)$ symmetry and show that it is the exact ground state of a model Hamiltonian with nearest-neighbor interactions constructed with either the $SO(2n+1)$ projection operators or more generally the $SU(2)$ spin projection operators. Unlike the valence bond solid state, we find that the hidden topological order in this class of matrix product states can be fully identified and characterized by a set of nonlocal string order parameters. When $n=1$, the $SO(3)$ symmetric matrix product state is exactly the same as the $S=1$ valence bond solid state and the model Hamiltonian possesses a hidden $Z_2 \times Z_2$ topological symmetry [6–8]. When $n>1$, it will be shown that the $SO(2n+1)$ ground state possesses a hidden $(Z_2 \times Z_2)^n$ topological symmetry. The breaking of this hidden symmetry leads to 4^n degenerate ground states with disentangled edge states in an open chain system.

Let us start by considering a one-dimensional lattice system with $SO(2n+1)$ symmetry. Each lattice site contains $2n+1$ basis states $\{|n^a\rangle; a = 1, \ldots, 2n+1\}$, which can be rotated within the $SO(2n+1)$ space as follows:

$$L^{ab}|n^c\rangle = \mathrm{i}\delta_{bc}|n^a\rangle - \mathrm{i}\delta_{ac}|n^b\rangle, \tag{1}$$

where $L^{ab} (a < b)$ are the $(2n^2+n)$ generators of the $SO(2n+1)$ Lie algebra, satisfying the following commutation relations:

$$[L^{ab}, L^{cd}] = \mathrm{i}(\delta_{ad}L^{bc} + \delta_{bc}L^{ad} - \delta_{ac}L^{bd} - \delta_{bd}L^{ac}). \tag{2}$$

According to the Lie algebra, the product of any two $SO(2n+1)$ vectors can be decomposed as a sum of an $SO(2n+1)$ scalar $\underline{1}$, an antisymmetric $SO(2n+1)$ tensor $\underline{2n^2+n}$, and a symmetric $SO(2n+1)$ tensor $\underline{2n^2+3n}$,

$$\underline{2n+1} \otimes \underline{2n+1} = \underline{1} \oplus \underline{2n^2+n} \oplus \underline{2n^2+3n}. \tag{3}$$

The number above each underline is the dimension of the irreducible representation.

In the spinor representation, the $SO(2n+1)$ generators can be expressed as $\Gamma^{ab} = [\Gamma^a, \Gamma^b]/2\mathrm{i}$, where Γ^a $(a = 1 \sim 2n+1)$ are the $2^n \times 2^n$ matrices that satisfy the Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}$ [10]. For each lattice site i, if the following matrix state is introduced:

$$g_i = \sum_a \Gamma^a |n^a\rangle_i,$$

then it can be readily shown that the bond product of g_i at any two neighboring sites have finite projection only in the scalar $\underline{1}$ and the antisymmetric $\underline{2n^2+n}$ subspaces spanned by $|n_i^a\rangle$ and $|n_{i+1}^a\rangle$ states, because the product of Γ^a and Γ^b can be expressed as $\Gamma^a\Gamma^b = \delta_{ab} + \mathrm{i}\Gamma^{ab}$. This is a special property of the $SO(2n+1)$ spinor representation constructed by Clifford algebra. By applying this argument to a periodic chain, we can show that the matrix product state defined by

$$|\Psi\rangle = \mathrm{Tr}(g_1 g_2 \cdots g_L), \tag{4}$$

is the exact ground state of the following $SO(2n+1)$ symmetric Hamiltonian:

$$H_{SO(2n+1)} = \sum_i \mathcal{P}_{\underline{2n^2+3n}}(i, i+1), \tag{5}$$

where $\mathcal{P}_{\underline{2n^2+3n}}(i, j)$ is a projection operator that projects the states at sites i and j onto their $SO(2n+1)$ symmetric tensor $\underline{2n^2+3n}$. To compute the static correlation functions of the matrix product ground state (4), we can use a transfer matrix method [8, 11]. At large distance, the two-point correlation functions of $SO(2n+1)$ generators decay exponentially as

$$\langle L_i^{ab} L_j^{ab}\rangle \sim \exp\left(-\frac{|j-i|}{\xi}\right), \tag{6}$$

with the correlation length $\xi = 1/\ln\left|\frac{2n+1}{2n-3}\right|$.

For the three $SO(2n+1)$ channels given in equation (3), the bond Casimir charge $\sum_{a<b}\left(L_i^{ab}+L_j^{ab}\right)^2$ for two adjacent sites takes the values 0, $4n-2$ and $4n+2$, respectively. Combining this result with the equation $\sum_{a<b}\left(L_i^{ab}\right)^2 = 2n$ and the completeness condition of the projection operators, we can then express the bond projection operator $\mathcal{P}_{\underline{2n^2+3n}}(i,j)$ with the $SO(2n+1)$ generators as

$$\mathcal{P}_{\underline{2n^2+3n}}(i,j) = \frac{1}{2}\sum_{a<b} L_i^{ab}L_j^{ab} + \frac{1}{4n+2}\left(\sum_{a<b} L_i^{ab}L_j^{ab}\right)^2 + \frac{n}{2n+1}.$$

Thus the model defined by equation (5) is a bilinear–biquadratic Hamiltonian in terms of the $SO(2n+1)$ generators.

At each lattice site, the $2n+1$ vectors of $SO(2n+1)$ can be also constructed from the $S=n$ quantum spin states. In the $SU(2)$ spin language, the last two channels in equation (3) correspond to the total bond spin $S=1,3,\ldots,2n-1$ and $S=2,4,\ldots,2n$ states, respectively. Furthermore, it can be shown that the bond projection operators of $SO(2n+1)$ can be expressed using the spin projection operators $P_{S=m}(i,j)$ as

$$\mathcal{P}_{\underline{2n^2+n}}(i,j) = \sum_{m=1}^{n} P_{S=2m-1}(i,j), \qquad \mathcal{P}_{\underline{2n^2+3n}}(i,j) = \sum_{m=1}^{n} P_{S=2m}(i,j).$$

Thus $\mathcal{P}_{\underline{2n^2+3n}}(i,j)$ is to project the spin states at sites i and j onto the nonzero even total spin states. Based on this property, we can further show that the matrix product wavefunction (4) is also the ground state of the following integer spin Hamiltonian:

$$H_{SU(2)} = \sum_i \sum_{m=1}^{n} J_m P_{S=2m}(i,i+1) \tag{7}$$

with all $J_m > 0$. This model is $SU(2)$-invariant in general. However, the ground state (4) possesses an *emergent* $SO(2n+1)$ symmetry. When all $J_m = 1$, $H_{SU(2)}$ becomes $SO(2n+1)$-invariant. In this case, $H_{SU(2)}$ simply reduces to $H_{SO(2n+1)}$.

It is interesting to compare $H_{SU(2)}$ with the AKLT model of valence bond solid proposed by Affleck *et al* [2, 3],

$$H_{\text{AKLT}} = \sum_i \sum_{m=n+1}^{2n} K_m P_{S=m}(i,i+1) \tag{8}$$

with all $K_m > 0$. The ground state of H_{AKLT} is also a matrix product state similar to equation (4), but g_i is now a $(S+1)\times(S+1) = (n+1)\times(n+1)$ matrix [8]. These two matrix product states have different topological properties and belong to different topological phases when $n>1$. Therefore $H_{SU(2)}$ and $H_{SO(2n+1)}$ can be viewed as a new family of exactly solvable quantum integer spin models to understand the internal structures of Haldane gap phases.

When $n=1$, both $H_{SO(2n+1)}$ and $H_{SU(2)}$ become exactly the same as the $S=1$ AKLT model H_{AKLT}. The ground state has a hidden antiferromagnetic order in which the up and down spins lie alternately along the lattice, sandwiched by arbitrary number of non-polarized spin states. This dilute antiferromagnetic order can be measured by a nonlocal string order parameter first proposed by den Nijs and Rommelse [5],

$$\mathcal{O}^{\mu} = \lim_{|j-i|\to\infty}\left\langle S_i^{\mu}\prod_{l=i}^{j-1} e^{i\pi S_l^{\mu}} S_j^{\mu}\right\rangle = \frac{4}{9}, \tag{9}$$

where $\mu = x, y$ or z. By performing a nonlocal unitary transformation [6–8] to the spin operators with the following unitary operators:

$$U = \prod_{j<i} \exp\left(\mathrm{i}\pi S_j^z S_i^x\right), \tag{10}$$

two of the above string order parameters are converted into the conventional spin–spin correlation functions. The $SU(2)$ symmetry of the AKLT model is then reduced to a discrete $Z_2 \times Z_2$ symmetry [6–8]. This reveals a hidden topological symmetry of the original model. The breaking of this topological symmetry leads to the opening of the Haldane gap and the four-fold degenerate ground states in an open chain.

Similar to the $n = 1$ case, the general $SO(2n+1)$ $(n > 1)$ matrix product state (4) also contains interesting hidden antiferromagnetic orders. Since $SO(2n+1)$ is a rank-n algebra, one can always classify the states at each site using n quantum numbers (weights) $\{m_1, \ldots, m_n\}$ subjected to the constraint

$$m_\alpha m_\beta = 0, \qquad (\alpha \neq \beta). \tag{11}$$

Here $\{m_1, \ldots, m_n\}$ are the eigenvalues of the mutually commuting Cartan generators $\{L^{12}, L^{34}, \ldots, L^{2n-1,2n}\}$,

$$L^{2\alpha-1,2\alpha}|m_\alpha\rangle = m_\alpha|m_\alpha\rangle, \qquad (m_\alpha = 0, \pm 1). \tag{12}$$

According to equation (1), all these Cartan generators annihilate the state $\left|n^{2n+1}\right\rangle = |0, 0, \ldots, 0\rangle$. The other basis states are given by

$$|0\cdots, m_\alpha = \pm 1, \ldots 0\rangle = \tfrac{1}{\sqrt{2}}(|n^{2\alpha}\rangle \pm \mathrm{i}|n^{2\alpha-1}\rangle). \tag{13}$$

From the property of the Clifford algebra, the hidden antiferromagnetic order of the ground state $|\Psi\rangle$ can now be identified. In any of these $m_\alpha(\alpha = 1 \sim n)$ channel, it can be shown that $|m_\alpha\rangle$ is dilute antiferromagnetically ordered, same as for the $S = 1$ valence bond solid. Namely, the states of $m_\alpha = 1$ and $m_\alpha = -1$ will alternate in space if all the $m_\alpha = 0$ states between them are ignored. For example, a typical configuration of the ground state of the $SO(5)$ system is

$$\begin{array}{lcccccccccccccccc} m_1: & \cdots & 0 & \uparrow & 0 & 0 & \downarrow & \uparrow & 0 & 0 & 0 & \downarrow & \uparrow & 0 & \downarrow & 0 & \uparrow & \cdots \\ m_2: & \cdots & \uparrow & 0 & \downarrow & 0 & 0 & 0 & \uparrow & \downarrow & 0 & 0 & 0 & \uparrow & 0 & \downarrow & 0 & \cdots, \end{array}$$

where $(\uparrow, 0, \downarrow)$ represent $|m\rangle = (|1\rangle, |0\rangle, |-1\rangle)$ states, respectively.

This hidden antiferromagnetic order reminds us a generalization of the den Nijs–Rommelse nonlocal string order parameters to characterize this state. Similar to equation (9) of the $n = 1$ case [5], the string order parameters can be defined as

$$\mathcal{O}^{ab} = \lim_{|j-i|\to\infty} \left\langle L_i^{ab} \prod_{l=i}^{j-1} \exp\left(i\pi L_l^{ab}\right) L_j^{ab} \right\rangle. \tag{14}$$

Since the ground state is $SO(2n+1)$ rotationally invariant, the above nonlocal order parameters should all be equal to each other. Thus to determine the value of these parameters, only the value of $\mathcal{O}^{12}$ needs to be evaluated. In the L^{12} channel, the role of the phase factor in equation (14) is to correlate the finite spin polarized states in the m_1 channel at the two ends of the string. If nonzero m_1 takes the same value at the two ends, then the phase factor is equal to 1. On the other hand, if nonzero m_1 takes two different values at the two ends, then the phase factor is equal to -1. Thus the value of $\mathcal{O}^{12}$ is determined purely by the probability of $m_1 = \pm 1$ appearing at the two ends of the string. Since the ground state is translation invariant, it is straightforward to show that the probability of the states $m_1 = \pm 1$ appearing at one lattice site is $2/(2n+1)$ and thus $\mathcal{O}^{12} = 4/(2n+1)^2$.

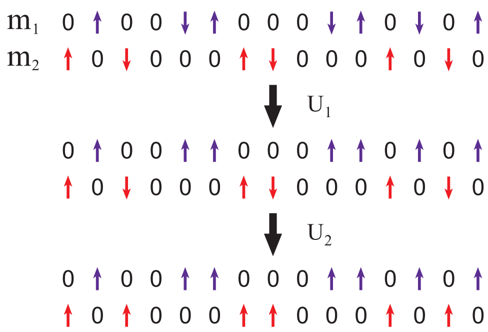

Figure 1. Changes of a typical configuration of the $SO(5)$ ground state under the unitary transformation defined by equation (16). U_1 and U_2 transform successively all m_1 and m_2 states to two diluted ferromagnetic configurations, respectively.

(This figure is in colour only in the electronic version)

The Kennedy–Tasaki unitary transformations (10) for $n = 1$ case [6–8] can also be generalized to arbitrary $n > 1$ cases. In the $SO(2n+1)$ Lie algebra, $(L^{2\alpha-1,2\alpha}, L^{2\alpha-1,2n+1}, L^{2\alpha,2n+1})$ span an $SO(3)$ sub-algebra in which $\exp(\mathrm{i}\pi L^{2\alpha,2n+1})$ plays the role of flipping the quantum number m_α. This exponential operator can flip the quantum numbers of m_α without disturbing the quantum states in all other channels. This indicates that if we take the following nonlocal unitary transformation in the m_α channel:

$$U_\alpha = \prod_{j<i} \exp\left(\mathrm{i}\pi L_j^{2\alpha-1,2\alpha} L_i^{2\alpha,2n+1}\right), \tag{15}$$

then all the configurations in this channel will be ferromagnetically ordered. Furthermore, by performing this nonlocal transformation successively in all the channels

$$U = \prod_{\alpha=1}^{n} U_\alpha, \tag{16}$$

then all the configurations of the ground state will become ferromagnetically ordered. As an example, figure 1 shows how the $SO(5)$ matrix product state $|\Psi\rangle$ is successively changed under this nonlocal unitary transformation.

By applying the generalized unitary transformation (16) to the *Cartan* generators, it can be shown that

$$U L_i^{ab} U^{-1} = L_i^{ab} \exp\left(\mathrm{i}\pi \sum_{j=1}^{i-1} L_j^{ab}\right). \tag{17}$$

Substituting this formula into equation (14), we find that

$$\mathcal{O}^{ab} = \lim_{|j-i|\to\infty} \left\langle L_i^{ab} L_j^{ab} \right\rangle_U. \tag{18}$$

Thus the nonlocal string order parameters $\mathcal{O}^{ab}$ become the ordinary correlation functions of local operators after the unitary transformation.

Under the above transformation, the symmetry of the original Hamiltonian $H_{SO(2n+1)}$ is reduced, and determined by the symmetry of the unitary transformation operators. In the m_α channel, it can be shown that the unitary operator U_α possesses only a $Z_2 \times Z_2$ symmetry [6–8]. Therefore, the Hamiltonian after the transformation has a $(Z_2 \times Z_2)^n$ symmetry. This is the hidden topological symmetry of the original Hamiltonian $H_{SO(2n+1)}$ associated with the hidden topological order of the original matrix product state $|\Psi\rangle$. Furthermore, the unitary transformation (16) breaks the translational symmetry. When it is applied to an open chain system, the hidden $(Z_2\times Z_2)^n$ topological symmetry of the Hamiltonian will be further broken, yielding 2^n free edge states at each end of the chain. Therefore, the open chain has totally 4^n degenerate ground states, which can be distinguished by their edge states.

As already mentioned, $H_{SO(2n+1)}$ is a bilinear–biquadratic Hamiltonian in terms of the $SO(2n+1)$ generators. Actually, we can introduce a general one-parameter family of the $SO(2n+1)$ bilinear–biquadratic model as

$$H=\sum_i\left[\cos\theta\sum_{a<b}L_i^{ab}L_{i+1}^{ab}+\sin\theta\left(\sum_{a<b}L_i^{ab}L_{i+1}^{ab}\right)^2\right],\tag{19}$$

which is an extension of the quantum spin-1 bilinear–biquadratic model. To determine the region of the Haldane gapped phase, we need to identity several special integrable points. At $\theta_1=\tan^{-1}\frac{1}{2n-1}$, the model (19) becomes the Uimin–Lai–Sutherland (ULS) model with an enhanced $SU(2n+1)$ symmetry, which can be solved by Bethe ansatz [12]. It is well known that this model has gapless excitations described by $SU(2n+1)_1$ Wess–Zumino–Witten model [13]. Based on the renormalization group approach, for $\theta<\theta_1$, Itoi and Kato [14] found that the marginally relevant interaction generates the Haldane gap, and the transition at the ULS point belongs to the universality class of the Kosterlitz–Thouless phase transition.

One the other hand, using quantum inverse scattering methods, Reshetikhin [15] had discovered another class of one-dimensional quantum integrable $SO(n)$ model, corresponding to the point $\theta_2=\tan^{-1}\frac{2n-3}{(2n-1)^2}$, where there are also gapless excitations above the ground state. For $n=1$, this point corresponds to the quantum spin-1 Takhatajan–Babujian model [16], which is at the boundary between Haldane gap phase and dimerized phase. These rigorous results suggest that the Haldane gapped phase for the general model (19) exists in the region

$$\tan^{-1}\frac{2n-3}{(2n-1)^2}<\theta<\tan^{-1}\frac{1}{2n-1}.\tag{20}$$

The exactly soluble point $\theta_{MPS}=\tan^{-1}\frac{1}{2n+1}$ has been included. In the whole region, we expect that the system has an energy gap in the excitations and the ordinary correlation functions display exponentially decay. However, a nonvanishing string order parameter (14) can measure the breaking of the hidden topological symmetry.

For $n=1$, the spin-1 quantum antiferromagnetic Heisenberg model ($\theta=0$) is just included in this region, however, we find that the $SO(2n+1)$ Heisenberg point for $n\geqslant 2$ does not belong to the Haldane gap phase. In particular, when $n=2$, the corresponding $SO(5)$ antiferromagnetic Heisenberg model has been used by Scalapino *et al* [17] to describe the $SO(5)$ 'superspin' phase on a ladder system of interacting electrons. Therefore, the ground-state and low-lying excitations of the quantum $SO(2n+1)$ symmetric generalized Heisenberg model for $n\geqslant 2$ deserves further studies.

In conclusion, we have constructed an $SO(2n+1)$-invariant matrix product state and shown that it is the exact ground state of an $SO(2n+1)$-symmetric Hamiltonian defined by equation (5) or more generally an $SU(2)$-symmetric spin Hamiltonian defined by equation (7). This matrix product state contains diluted antiferromagnetic orders in n different channels and a hidden $(Z_2\times Z_2)^n$ topological symmetry. These topological long-range order can be characterized by a set of nonlocal string order parameters. The breaking of the $(Z_2\times Z_2)^n$ topological symmetry leads to the opening of an excitation gap between the ground state and the first excitation state. In an open chain system, the 4^n edge states become completely disentangled and the ground states are 4^n degenerate. The multiple Z_2 nature of these topological states suggests that they can serve as a resource of multiple qubits. We believe that these states, similar as for the $S = 1$ AKLT valence bond state, can be encoded to perform ideal quantum teleportation [18] or fault-tolerant quantum computation through local spin measurements.

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References

[1] Haldane F D M 1983 *Phys. Lett.* A **93** 464
Haldane F D M 1983 *Phys. Rev. Lett.* **50** 1153
[2] Affleck I, Kennedy T, Lieb E H and Tasaki H 1987 *Phys. Rev. Lett.* **59** 799
Affleck I, Kennedy T, Lieb E H and Tasaki H 1988 *Commun. Math. Phys.* **115** 477
[3] Arovas D P, Auerbach A and Haldane F D M 1988 *Phys. Rev. Lett.* **60** 531
[4] Girvin S M and Arovas D P 1989 *Phys. Scr.* T **27** 156
[5] den Nijs M and Rommelse K 1989 *Phys. Rev.* B **40** 4709
[6] Kennedy T and Tasaki H 1992 *Phys. Rev.* B **45** 304
Kennedy T and Tasaki H 1992 *Commun. Math. Phys.* **147** 431
[7] Oshikawa M 1992 *J. Phys.: Condens. Matter* **4** 7469
[8] Totsuka K and Suzuki M 1995 *J. Phys.: Condens. Matter* **7** 1639
[9] Tu H H, Zhang G M and Xiang T 2008 arXiv:0807.3143
[10] Georgi H 1999 *Lie algebras in Particle Physics* (Reading, MA: Perseus Books)
[11] Klümper A, Schadschneider A and Zittartz J 1991 *J. Phys. A: Math. Gen.* **24** L955
Klümper A, Schadschneider A and Zittartz J 1992 *Z. Phys. B: Condens. Matter* **87** 281
[12] Uimin G V 1970 *JETP Lett.* **12** 225
Lai C K 1974 *J. Math. Phys.* **15** 1675
Sutherland B 1975 *Phys. Rev.* B **12** 3795
[13] Affleck I 1986 *Nucl. Phys.* B **265** 409
[14] Itoi C and Kato M H 1997 *Phys. Rev.* B **55** 8295
[15] Reshetikhin N Y 1983 *Lett. Math. Phys.* **7** 205
Reshetikhin N Y 1985 *Theor. Math. Phys.* **63** 555
[16] Takhatajan L A 1982 *Phys. Lett.* A **87** 479
Babujian H M 1982 *Phys. Lett.* A **90** 479
[17] Scalapino D, Zhang S C and Hanke W 1998 *Phys. Rev.* B **58** 443
[18] Verstraete F, Martin-Delgado M A and Cirac J I 2004 *Phys. Rev. Lett.* **92** 087201